Goal: Define and understand $\mathrm{cost}(\Omega)$

- $\mathrm{cost}(\Omega) \leq \mathrm{cost}(\Phi(\Omega))$
- $(\mathbb{M}_0, R, \mu_0)$ is p.m.p. ("Palm meas")

$\mathbb{M}_0 = \{\omega \in \mathbb{M} \mid 0 \in \omega\}$ "rooted" config.

Take R to be O.E. of $G \curvearrowright \mathbb{M}$ restricted to $\mathbb{M}_0$

"Rerooting": Declare $\omega \in \mathbb{M}_0$ has $\omega \sim_R g^{-1}\omega$ $\forall g \in \omega$. ($0 \in g^{-1}\omega$)

$$\left\{\begin{matrix}\text{Borel directed factor graphs} \\ \mathcal{G} : \mathbb{M} \to \mathrm{Graph}(G)\end{matrix}\right\} \xleftrightarrow{1-1} \left\{\text{Borel subsets of } \overrightarrow{\mathbb{M}_0}\right\}$$

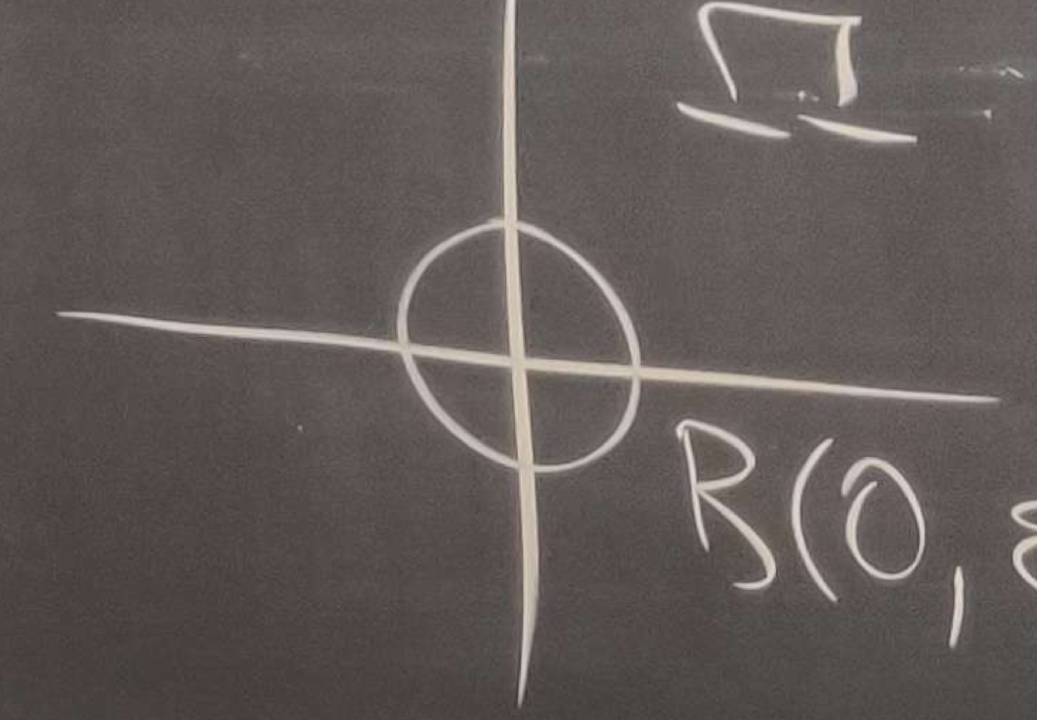
$\overrightarrow{\mathbb{M}_0} = \{(\omega, g) \in \mathbb{M}_0 \times G \mid$

Given $\mathcal{G}$ directed

Let $A_{\mathcal{G}} = \{(\omega, g) \in \overrightarrow{\mathbb{M}_0} \mid$

Given $A \subseteq \overrightarrow{\mathbb{M}_0}$, let

The Palm process of



For

st(Π)

$_0$) is pmp.
"Palm mecs."

cted to $\mathbb{M}_0$

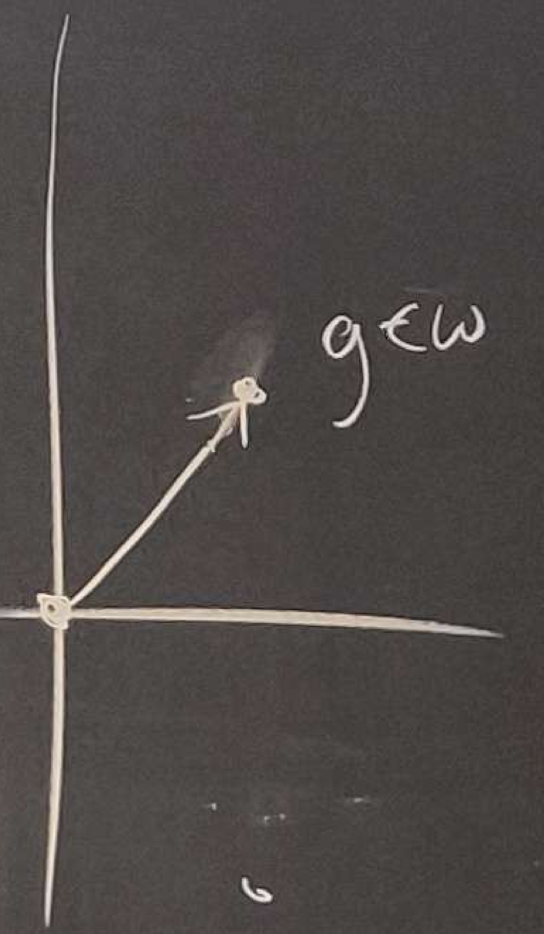
$\omega \ \forall g\in\omega$.
$0 \in g^{-1}\omega$

of $\overrightarrow{\mathbb{M}_0}$ }

$\overrightarrow{\mathbb{M}_0} = \{(\omega, g)\in \mathbb{M}_0\times G \mid g\in\omega\}$

Given $\mathcal{G}$ directed factor graph.

Let $A_{\mathcal{G}} = \{(\omega, g)\in \overrightarrow{\mathbb{M}_0} \mid 0 \to g \text{ in } \mathcal{G}(\omega)\}$



Given $A \subseteq \overrightarrow{\mathbb{M}_0}$, let $\mathcal{G}_A(\omega) = \{(g,h)\in\omega\times\omega \mid (g^{-1}\omega, g^{-1}h)\in A\}$

The Palm process of Π is Π_0 — "Π conditioned on $\mathbb{M}_0$"

For Π Poisson, $\Pi_0 \overset{(d)}{=} \Pi \cup \{0\}$.

Π has law μ. Want to d

Given $A \subseteq \mathbb{M}_0$, define the

$\theta_A(\Pi) = \{g\in\Pi \mid g^{-1}\Pi \in A$

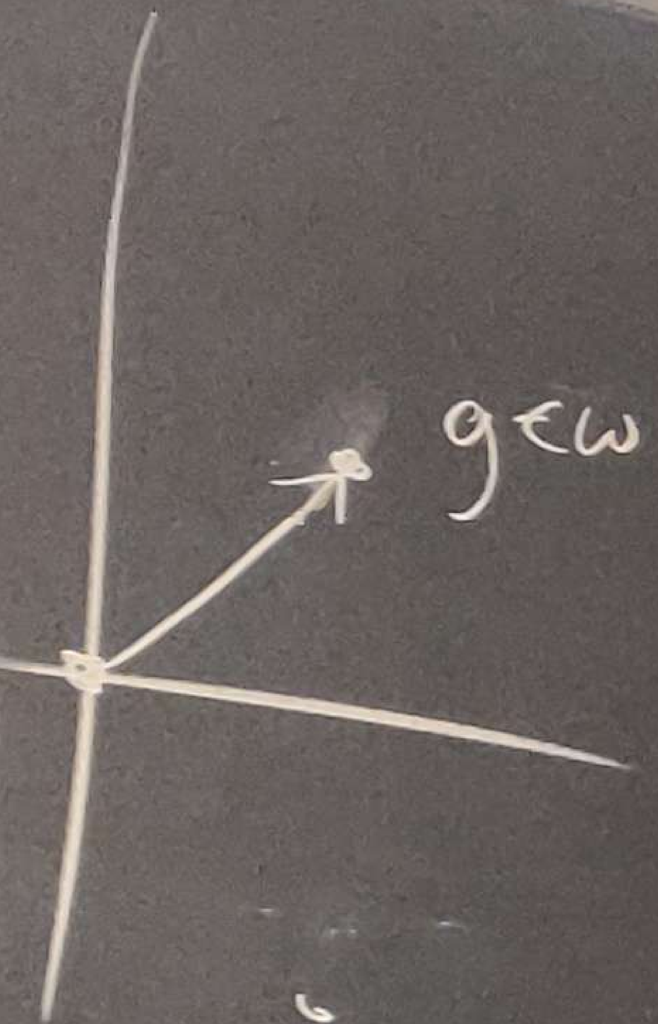
Let $\mu_0(A) = \dfrac{\text{"int. } \theta_A(\Pi)\text{"}}{\text{int}(\Pi)} = $ in

where $\lambda(U) = 1$.

We say Π_0 is a Palm ve

$\mathbb{P}[\Pi_0 \in A] = \mu_0(A)$.

$l,$.



$(\omega)\}$

$h) \in \omega \times \omega \mid (g^{-1}\omega, g^{-1}h) \in A\}$

— "Ω conditioned on $\mathbb{M}_0$"

$n, \quad \Omega_0 \overset{(d)}{=} \Omega \cup \{0\}.$

Ω has law μ. Want to define $\mu_0 \in \mathrm{Prob}(\mathbb{M}_0)$

Given $A \subseteq \mathbb{M}_0$, define the "A points" of Ω as

$$\Theta_A(\Omega) = \{g \in \Omega \mid g^{-1}\Omega \in A\} \subseteq \Omega$$

Let $$\mu_0(A) = \frac{\mathrm{int}.\,\Theta_A(\Omega)}{\mathrm{int}.(\Omega)} = \frac{1}{\mathrm{int}(\Omega)} \mathbb{E}\left[\#\{g \in U \mid g^{-1}\Omega \in A\}\right],$$

where $\lambda(U) = 1$.

We say Ω_0 is a Palm version of Ω if its law is μ_0,

$$\mathbb{P}[\Omega_0 \in A] = \mu_0(A).$$

[Holroyd-Peres] The Poisson on $\mathbb{R}^n$ admits $\mathbb{Z}$ as a connected factor graph.

[Timar] Every free + ergodic point process on $\mathbb{R}^n$ admits $\forall d$ $\mathbb{Z}^d$ as a connected factor graph

$\Gamma = \langle S \rangle$ $\quad \Omega$ P.P. on G

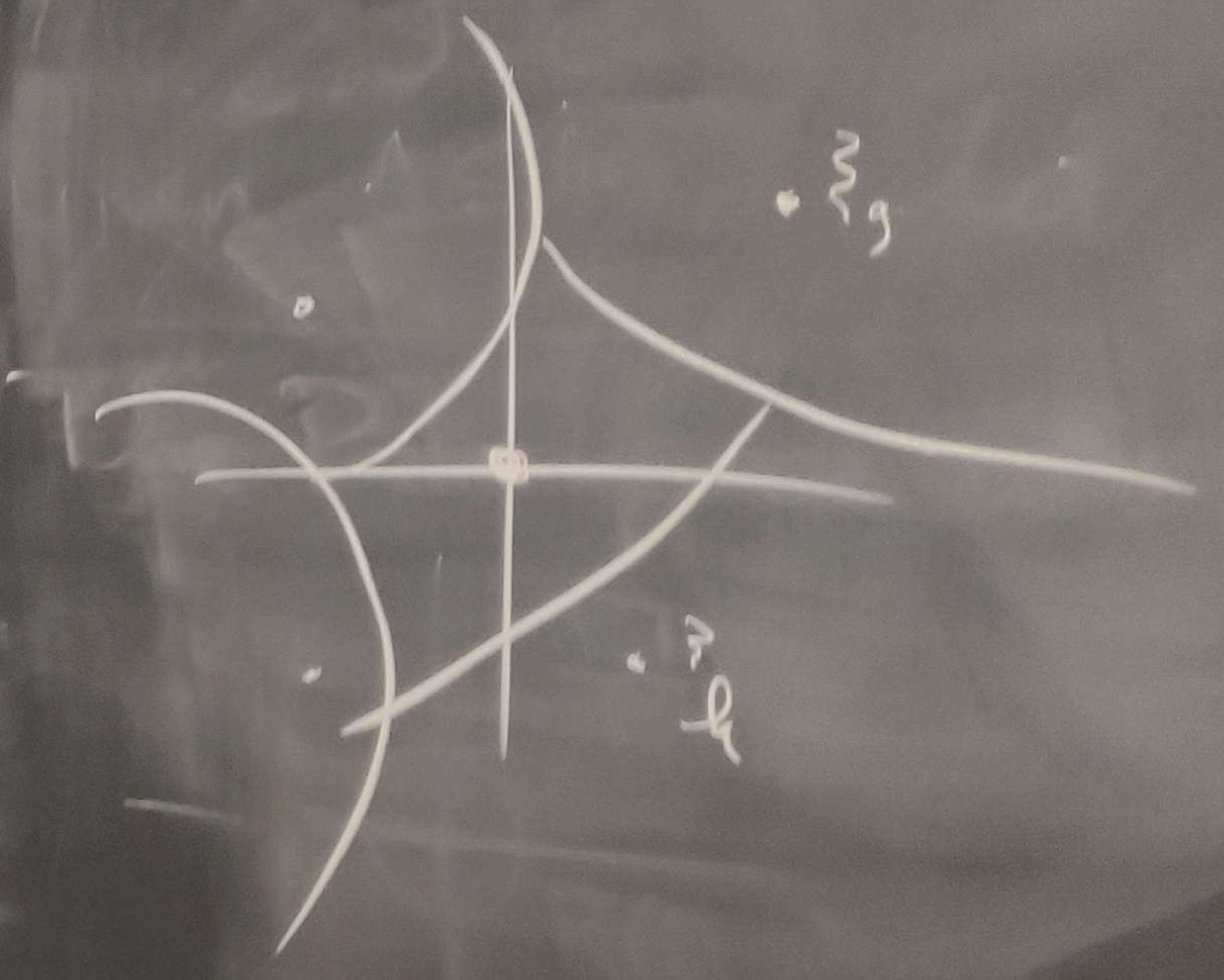
G admits $Cay(\Gamma, S)$ as a connected factor graph

$\Leftrightarrow \Gamma \curvearrowright (\mathbb{M}_0, \mu_0)$ freely generating R preserving μ_0.

G amenable, so $\exists$ inva

Induces a mean on (

$\nu_\omega : \ell^\infty([\omega]) \longrightarrow \mathbb{R}$



...n $\mathbb{R}^n$ admits $\mathbb{Z}$ as
...ctor graph.

...oint process on $\mathbb{R}^n$

...onnected factor graph

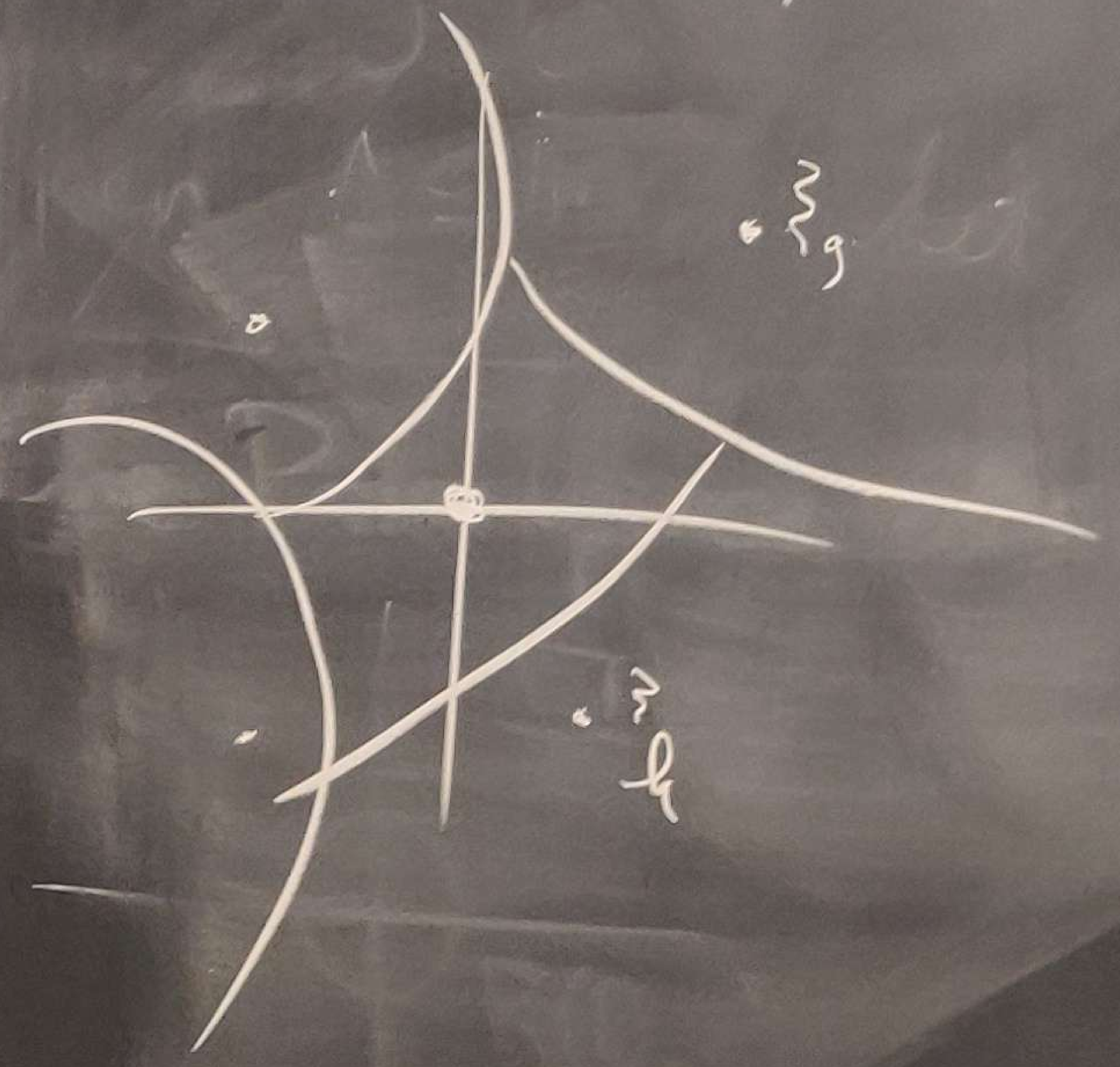
...d factor graph

...enerating R.

G amenable, so $\exists$ invariant mean $M : \ell^\infty(G) \to \mathbb{R}$.

Induces a mean on $(\mathbb{M}_0, R, \mu_0)$, need means

$\nu_\omega : \ell^\infty([\omega]) \longrightarrow \mathbb{R}$



Ω has law μ...

Given $A \subseteq \mathbb{M}_0$, ...

$\Theta_A(\Omega) = \{ g \in \Omega$ ...

Let $\mu_0(A) =$ int. Θ ... / int ...

where $\lambda(W) = 1$ ...

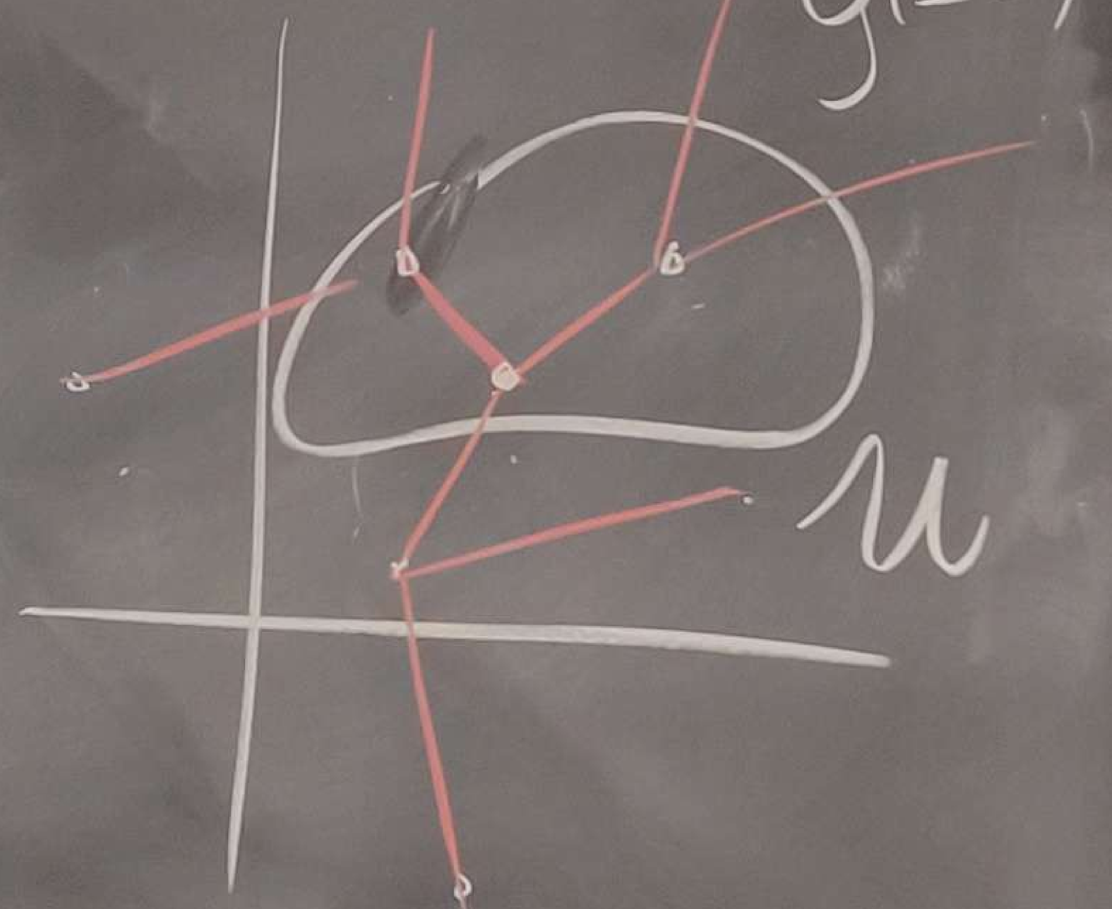
We say Ω_0 is ...

$\mathbb{P}[\Omega_0 \in A] = \mu$ ...

The average degree of Ω wrt factor graph $\mathcal{G}$

$$\mathrm{AvgDeg}(\mathcal{G}) = \frac{\mathbb{E}\left[\sum_{x \in \mathcal{U}} \deg_x(\mathcal{G}(\Omega))\right]}{\mathbb{E}[N_{\mathcal{U}}(\Omega)]}$$

$$= \mathrm{int}(\Omega) \cdot \lambda(\mathcal{U})$$



$$\mathrm{AvgDeg}(\mathcal{G}) = \mathbb{E}\left[\deg_0(\mathcal{G}(\Omega_0))\right]$$

$$\mathrm{cost}(\Omega) - 1 = \inf_{\mathcal{G}} \left\{ \frac{1}{2} A \ldots \right.$$

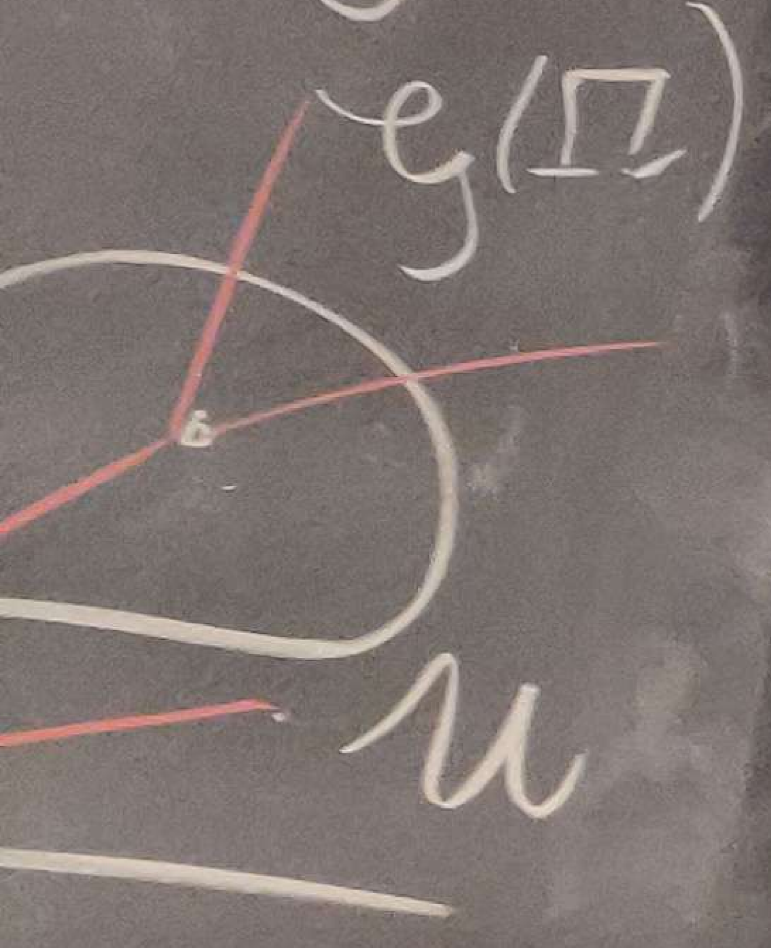
$$\lambda(\mathcal{U})\, \mathrm{int.}(\Omega) = \mathbb{E}[N_{\mathcal{U}}(\Omega)]$$

$$\mathrm{int.}(\Omega) \int_G \mathbb{1}[g \in \mathcal{U}]\, d\lambda(g) =$$

$$\rightsquigarrow \mathrm{int}(\Omega) \int_G f(g)\, d\lambda(g)$$

for $f : G \to \mathbb{R}$

h $\mathcal{G}$



$$\mathrm{cost}(\Pi) - 1 = \inf_{\mathcal{G}} \left\{ \frac{1}{2} \mathrm{AvgDeg}(\mathcal{G}) \right\} - \mathrm{intensity}(\Pi)$$

$$\lambda(U)\, \mathrm{int.}(\Pi) = \mathbb{E}[N_U(\Pi)]$$

$$\mathrm{int.}(\Pi) \int_G \mathbb{1}[g \in U]\, d\lambda(g) = \mathbb{E}\Big[\sum_{g \in \Pi} \mathbb{1}[g \in U]\Big]$$

$$\rightsquigarrow \mathrm{int}(\Pi) \int_G f(g)\, d\lambda(g) = \mathbb{E}\Big[\sum_{g \in \Pi} f(g)\Big]$$

for $f : G \to \mathbb{R}$.

intensity(Π)

$[q \in u]$

$f(q)$

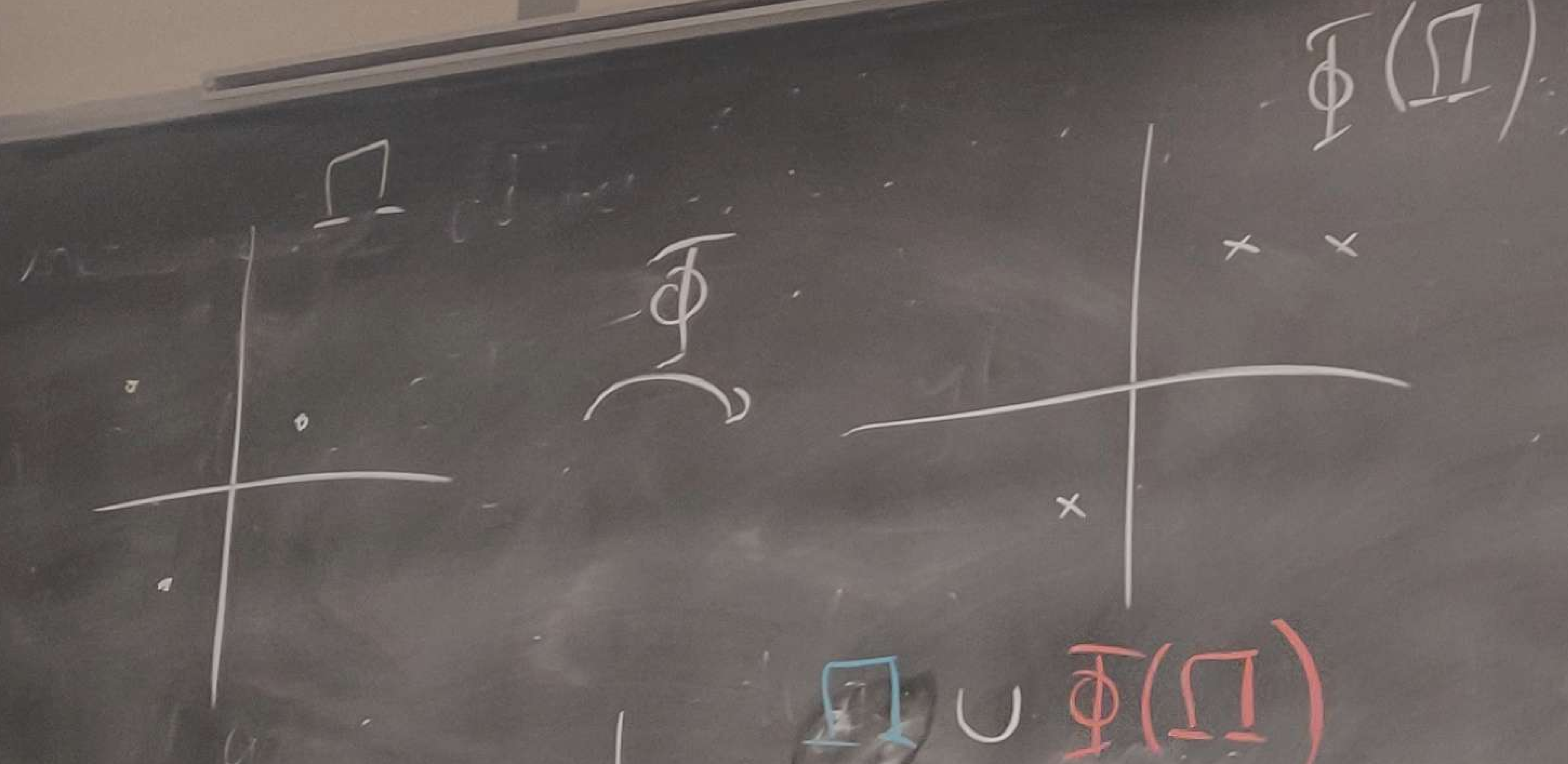
$\Phi : \mathbb{M} \to \mathbb{M}$ factor map

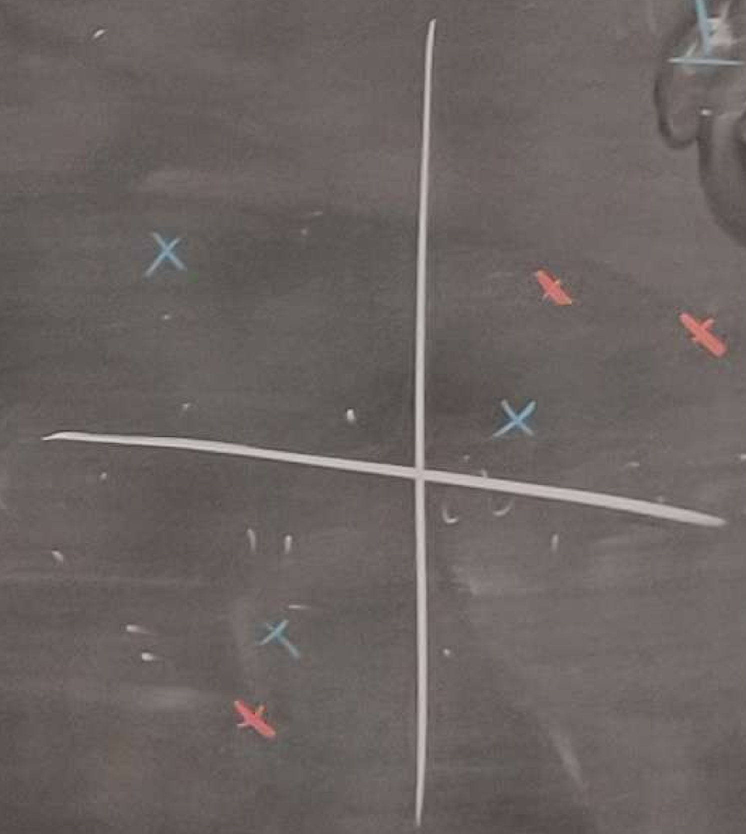
Φ can be written as a combination of a thinning a thickening and a colouring

· $\theta : \mathbb{M} \to \mathbb{M}$ is a thinning if $\theta(\omega) \subset \omega$ $\forall \omega$

· $\Theta : \mathbb{M} \to \mathbb{M}$ is a thickening if $\Theta(\omega) \supset \omega$ $\forall \omega$

· $\mathcal{C} : \mathbb{M} \to \Xi^{\mathbb{M}}$ is a colouring if the underlying set of $\mathcal{C}(\omega)$ is ω $\forall \omega$.





$\Omega \cup \Phi(\Omega)$ is a coloured thickening and can be thinned to $\Phi(\Omega)$

$$\mathrm{cost}(\Omega) - 1 = \inf_{g} \Big\{ \frac{1}{2} A \ldots$$

$$\lambda(U)\,\mathrm{int.}(\Omega) = \mathbb{E}[N_U(\Omega)]$$

$$\mathrm{int.}(\Omega) \int_G \mathbb{1}[g \in U]\, d\lambda(g) = \ldots$$

$$\rightsquigarrow \mathrm{int}(\Omega) \int_G f(g)\, d\lambda(g) \ldots$$

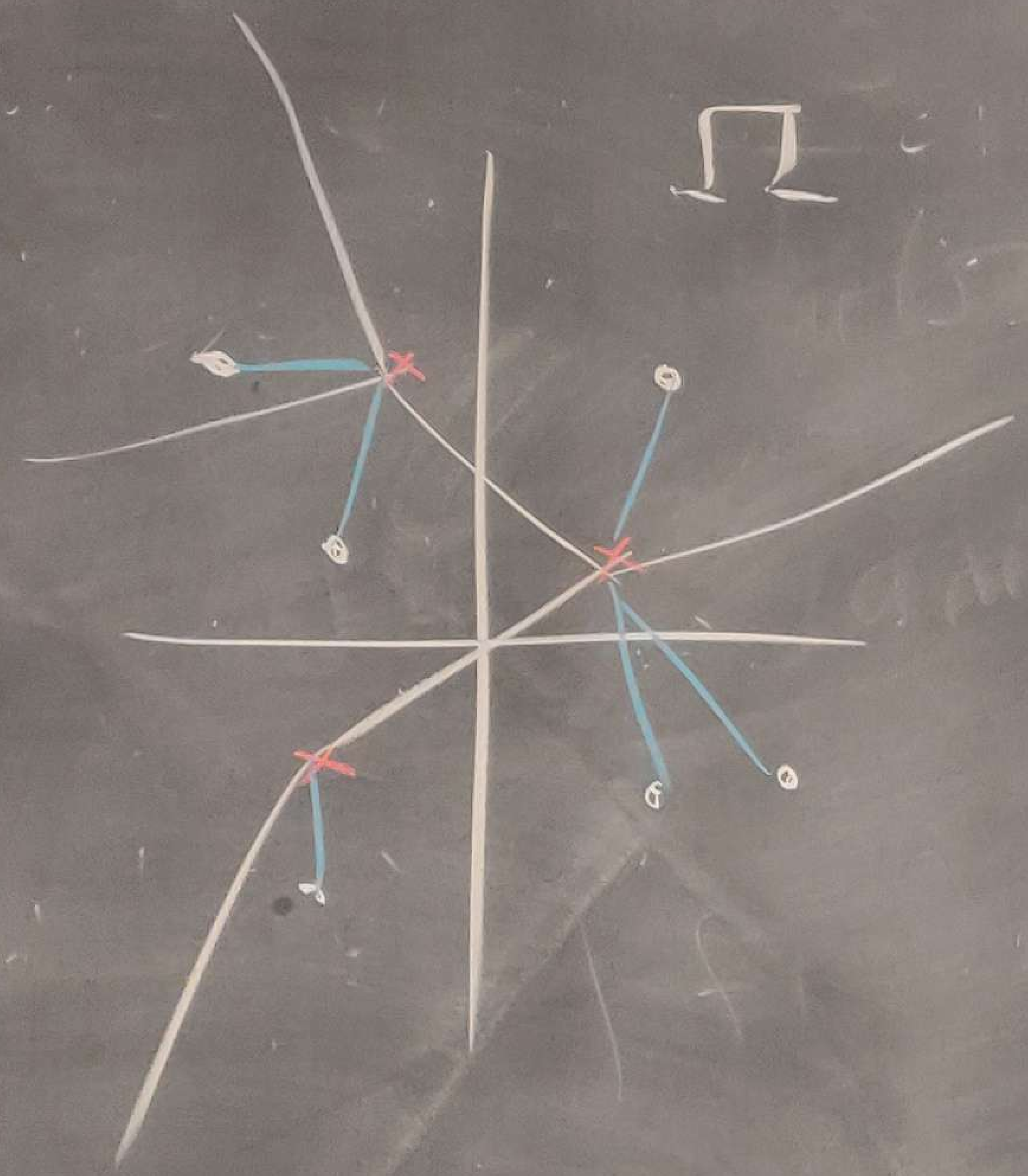
for $f : G \to \mathbb{R}$

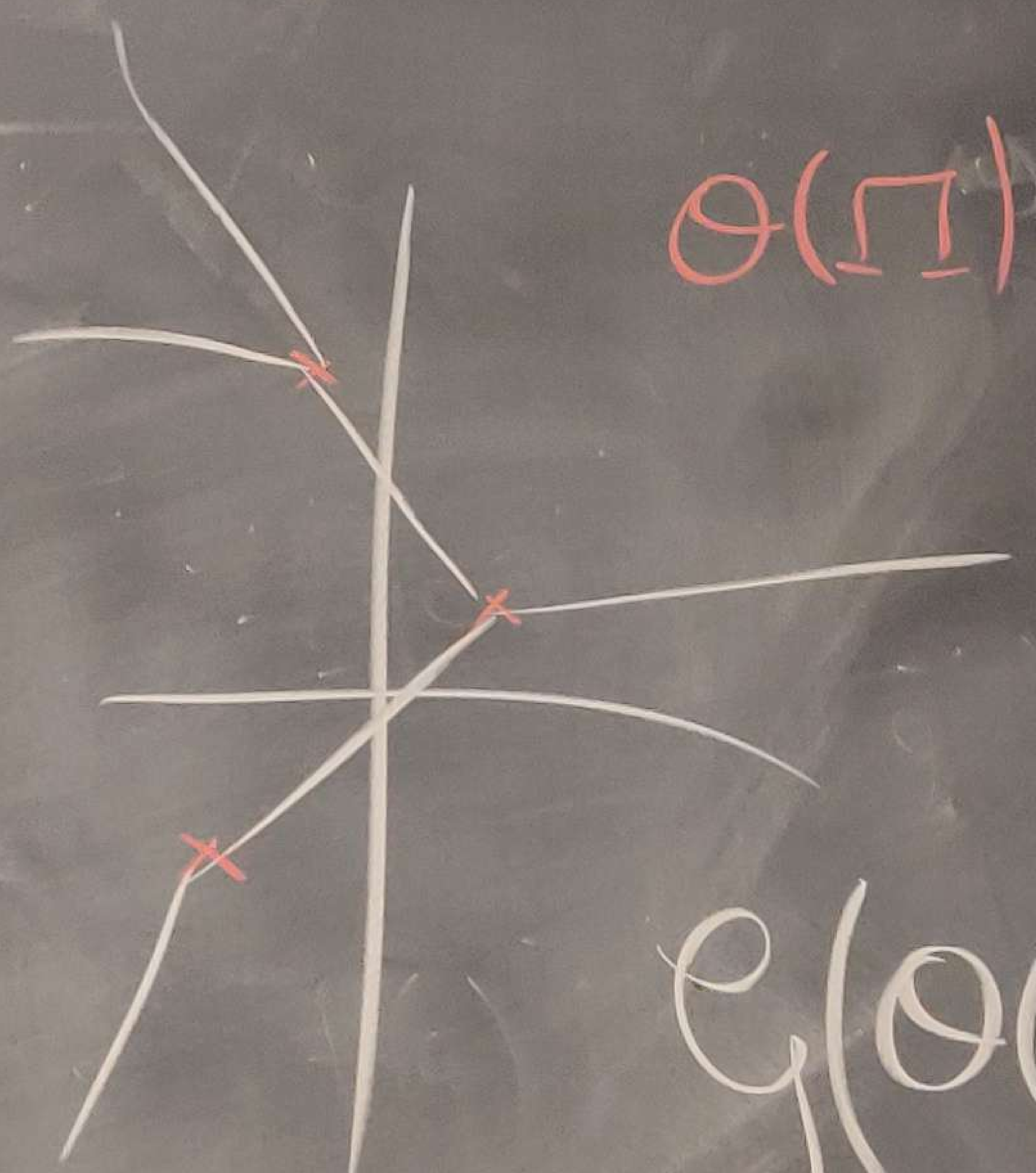
$\Phi(\Pi)$

ured thickening

be thinned to

$\text{cost}(\Pi) \le \text{cost}(\Theta(\Pi))$ thinning case





$\mathcal{G}(\Theta(\Pi))$

$\Phi : M \to$ ... fa...

Φ can be ...

case
$\Theta(\Pi)$
$\mathcal{G}(\Theta(\Pi))$
$\mathrm{Cost}(\Pi) \leq \mathrm{cost}(\Theta(\Pi))$
thickening case
$\Theta(\Pi)$
$\mathcal{G}(\Theta(\Pi))$
Voronoi of red pts